MATH 1010E University Mathematics
Lecture Notes (week 12)
Martin Li

1 Trigonometric Substitutions

Sometimes trigonometric functions can help us solve integrals that originally do not contain any trigonometric functions in them. Let us consider the following integral

$$\int \sqrt{1-x^2}\, dx,$$

if we use the u-substitution method by setting $x = \sin\theta$, then $dx = d(\sin\theta) = \cos\theta\, d\theta$ and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta.$$

Therefore, the integral becomes

$$\int \sqrt{1-x^2}\, dx = \int \cos\theta \cdot \cos\theta\, d\theta = \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C.$$

We can write the answer back in terms of x by $\theta = \sin^{-1} x$ and

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta} = 2x\sqrt{1-x^2}.$$

Therefore,

$$\int \sqrt{1-x^2}\, dx = \frac{1}{2}(\sin^{-1} x + x\sqrt{1-x^2}) + C.$$

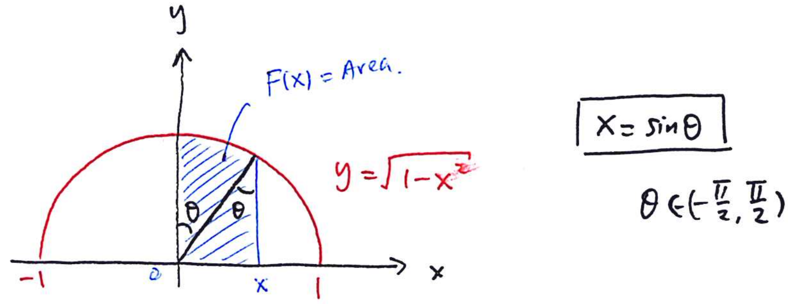
Question: What can we take $\cos\theta \geq 0$ so that $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ in this example?

In fact there is a more geometric way to solve this integral. Remember that the Fundamental Theorem of Calculus I says that

$$F(x) := \int_0^x \sqrt{1-t^2}\, dt$$

is a primitive function, i.e. $F'(x) = \sqrt{1-x^2}$.

Geometrically, $F(x)$ is a definite integral which computes the blue area below:



The blue region consists of a circular sector and a right angled triangle. Therefore, by elementary plane geometry,

$$
\begin{aligned}
F(x) &= \text{area of sector} + \text{area of triangle} \\
&= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} \\
&= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2}.
\end{aligned}
$$

This gives the same answer as before.

Similar ideas can help us solve a number of integrals involving $\sqrt{a^2-x^2}$, a^2+x^2 or $\sqrt{x^2-a^2}$.

Theorem 1.1 *Let $a>0$ be a positive constant. We have the following:*

(1) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C.$

(2) $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$

(3) $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C.$

Proof: (1) Let $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$ and

$$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2\cos^2\theta}} = \frac{1}{a\cos\theta}.$$

Therefore,

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\cos\theta}\cdot a\cos\theta\,d\theta = \theta + C = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

(2) Let $x = a\tan\theta$, then $dx = a\sec^2\theta\, d\theta$ and

$$\frac{1}{a^2+x^2} = \frac{1}{a^2+a^2\tan^2\theta} = \frac{1}{a^2\sec^2\theta}.$$

Therefore, the integral becomes

$$\int \frac{1}{a^2+x^2}\, dx = \int \frac{1}{a^2\sec^2\theta}\cdot a\sec^2\theta\, d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

(3) Let $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta\, d\theta$ and

$$\frac{1}{x\sqrt{x^2-a^2}} = \frac{1}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}} = \frac{1}{a^2\sec\theta\tan\theta}.$$

Therefore, the integral becomes

$$\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \int \frac{a\sec\theta\tan\theta\, d\theta}{a^2\sec\theta\tan\theta} = \frac{\theta}{a} + C = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C.$$

In summary, it is therefore suggestive (but not always) to use the following substitutions if we see expressions below:

$$a^2 - x^2 \quad \Leftrightarrow \quad x = a\sin\theta$$

$$a^2 + x^2 \quad \Leftrightarrow \quad x = a\tan\theta$$

$$x^2 - a^2 \quad \Leftrightarrow \quad x = a\sec\theta$$

Let us look at more examples below.

Example 1.2 *Consider the integral*

$$\int \frac{x^3}{\sqrt{4-x^2}}\, dx,$$

Let $x = 2\sin\theta$, then $dx = 2\cos\theta\, d\theta$ and

$$\frac{x^3}{\sqrt{4-x^2}} = \frac{8\sin^3\theta}{2\cos\theta}.$$

The integral becomes

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^2}}\,dx &= \int \frac{8\sin^3\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta \\
&= \int 8\sin^3\theta\,d\theta \\
&= 8\int \sin^2\theta(\sin\theta\,d\theta) \\
&= 8\int \sin^2\theta\,d(-\cos\theta) \\
&= -8\int (1-\cos^2\theta)\,d(\cos\theta) \\
&= -8\left(\cos\theta - \frac{\cos^3\theta}{3}\right) + C \\
&= -8\cos\theta\left(1 - \frac{\cos^2\theta}{3}\right) + C.
\end{aligned}$$

To rewrite it in terms of x, note that

$$\sin\theta = \frac{x}{2} \qquad \text{and} \qquad \cos\theta = \sqrt{1-\frac{x^2}{4}}.$$

Therefore,

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx = -8\sqrt{1-\frac{x^2}{4}}\left(1-\frac{1}{3}\left(1-\frac{x^2}{4}\right)\right) + C.$$

Example 1.3 *Consider the integral*

$$\int \sqrt{\frac{1+x}{1-x}}\,dx,$$

the integral as it is does not contain any a^2-x^2, a^2+x^2 nor x^2-a^2 term, but we can transform it to

$$\sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{(1+x)^2}{1-x^2}} = \frac{1+x}{\sqrt{1-x^2}}.$$

Therefore,

$$\begin{aligned}
\int \sqrt{\frac{1+x}{1-x}}\,dx &= \int \frac{1+x}{\sqrt{1-x^2}}\,dx \\
&= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx \\
&= \int \frac{1}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} \\
&= \sin^{-1} x - \sqrt{1-x^2} + C.
\end{aligned}$$

Example 1.4 *Consider the integral*

$$\int \frac{dx}{\sqrt{4+x^2}},$$

if we let $x = 2\tan\theta$, then $dx = 2\sec^2\theta\,d\theta$ and

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2\sec\theta}.$$

Therefore, the integral becomes

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Rewriting back in terms of x, we use

$$\tan\theta = \frac{x}{2} \qquad \text{and} \qquad \sec\theta = \sqrt{1+\frac{x^2}{4}},$$

therefore,

$$\int \frac{dx}{\sqrt{4+x^2}} = \ln\left|\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right| + C.$$

2 Reduction Formula

There is a useful technique called *reduction formula* that simplifies an integral in a systematic way. Let us look at the following example.

Example 2.1 *Consider the integral*

$$\int \cos^n x\,dx,$$

when $n = 0$,

$$\int \cos^0 x \, dx = \int 1 \, dx = x + C.$$

When $n = 1$,

$$\int \cos x \, dx = \sin x + C.$$

The question is then, do we have a general formula for the integral

$$I_n := \int \cos^n x \, dx$$

for a general positive integer $n \geq 1$?

The idea is that we would hope to express I_n in terms of some I_k where $k < n$. Then, we can get I_n for n large from our knowledge about I_n for n small. This can be usually achieved by integration by parts:

$$\begin{aligned}
I_n = \int \cos^n x \, dx &= \int \cos^{n-1} x (\cos x \, dx) \\
&= \int \cos^{n-1} x \, d(\sin x) \\
&= \sin x \cos^{n-1} x - \int \sin x \, d(\cos^{n-1} x) \\
&= \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx \\
&= \sin x \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\
&= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx \\
&= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n.
\end{aligned}$$

Hence, if we move the I_n term to the left hand side and then divide out the constant, we obtain the following *reduction formula*:

$$I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}.$$

This formula tells us that I_n is related to I_{n-2} in a given way. Therefore, when n is odd, the reduction goes like

$$I_n \to I_{n-2} \to I_{n-4} \to \cdots \to I_3 \to I_1 = \sin x + C;$$

when n is even, we get

$$I_n \to I_{n-2} \to I_{n-4} \to \cdots \to I_2 \to I_0 = x + C.$$

In other words, we can get a general formula for I_n by working backwards from the above chain. The general formula would be a bit complicated for this example. If we are looking at definite integrals instead, sometimes the formula is simpler. For example, since the extra term $\frac{1}{n}\sin x \cos^{n-1} x = 0$ when $x = 0$ or $\frac{\pi}{2}$, the reduction formula reads

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx = \frac{n-1}{n} I_{n-2}.$$

Applying this inductively, we get for n even,

$$\begin{aligned} I_n &= \frac{n-1}{n} I_{n-2} \\ &= \frac{n-1}{n}\frac{n-3}{n-2} I_{n-4} \\ &= \frac{n-1}{n}\frac{n-3}{n-2} \cdots \frac{3}{4}\frac{1}{2} I_0 \\ &= \frac{(n-1)(n-3)\cdots 3 \cdot 1}{n(n-2)\cdots 4 \cdot 2} \cdot \frac{\pi}{2}. \end{aligned}$$

Exercise: Work out the formula for n odd.

Example 2.2 *Let us look at a similar integral*

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx,$$

We can derive a reduction formula as the previous example and hence obtain a general formula for n. However, there is actually a much quicker method using change of variable. Recall that

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x.$$

Therefore, if we let $x = \frac{\pi}{2} - u$, then $dx = -du$ and when $x = 0$, $u = \frac{\pi}{2}$; when $x = \frac{\pi}{2}$, $u = 0$. Therefore,

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_{\frac{\pi}{2}}^0 \sin^n\left(\frac{\pi}{2} - u\right)(-du) = \int_0^{\frac{\pi}{2}} \cos^n u \, du,$$

which is the integral we have just studied above. Therefore, we have the same formula since

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx.$$

Example 2.3 *Derive a reduction formula for*

$$I_n = \int x^n e^{ax} \, dx, \qquad n \geq 0$$

where $a \in \mathbb{R}$ is a fixed constant.

Using integration by parts, we have

$$\begin{aligned} I_n = \int x^n e^{ax} \, dx &= \frac{1}{a} \int x^n \, d(e^{ax}) \\ &= \frac{1}{a} x^n e^{ax} - \frac{1}{a} \int e^{ax} \, d(x^n) \\ &= \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx \\ &= \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}. \end{aligned}$$

Therefore, the reduction formula is

$$I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}.$$

In this case, the parameter reduce by 1 only, so there is only one chain

$$I_n \to I_{n-1} \to \cdots \to I_3 \to I_2 \to I_1 \to I_0,$$

and we just need to compute I_0 explicitly to get all the I_n.
Challenging Exercise: Derive a reduction formula for

$$I_n = \int \left(\frac{\sin \frac{x-a}{2}}{\sin \frac{x+a}{2}} \right)^n dx, \qquad n \geq 1.$$

3 Integrals of Piecewise Defined Functions

Sometimes a continuous function is defined piecewise. We discuss how to integrate such functions in this section. Consider the following integral

$$\int |x| \, dx.$$

For $x > 0$, we have

$$\int |x|\, dx = \int x\, dx = \frac{1}{2}x^2 + C_1$$

and for $x < 0$, we have

$$\int |x|\, dx = \int -x\, dx = -\frac{1}{2}x^2 + C_2.$$

Note that C_1 and C_2 could be different constants. However, if we require the function

$$F(x) := \begin{cases} \frac{1}{2}x^2 + C_1 & \text{when } x > 0 \\ -\frac{1}{2}x^2 + C_2 & \text{when } x < 0 \end{cases}$$

to be continuous at $x = 0$, then we have to define $F(0) = C_1 = C_2$. Therefore, we have

$$\int |x|\, dx = F(x) := \begin{cases} \frac{1}{2}x^2 + C & \text{when } x \geq 0 \\ -\frac{1}{2}x^2 + C & \text{when } x < 0 \end{cases} \tag{3.1}$$

where C is ONE arbitrary constant.

For definite integrals, we just split up the integrals into subintervals where the function is defined by a single formula. For example,

$$\begin{aligned} \int_{-1}^{1} |x|\, dx &= \int_{-1}^{0} |x|\, dx + \int_{0}^{1} |x|\, dx \\ &= \int_{-1}^{0} -x\, dx + \int_{0}^{1} x\, dx \\ &= -\frac{1}{2}x^2\Big|_{-1}^{0} + \frac{1}{2}x^2\Big|_{0}^{1} \\ &= (0 + \frac{1}{2}) + (\frac{1}{2} - 0) = 1. \end{aligned}$$

We can also use (3.1) together with the fundamental theorem of calculus to evaluate the definite integrals:

$$\int_{-1}^{1} |x|\, dx = F(1) - F(-1) = \left(\frac{1}{2} + C\right) - \left(-\frac{1}{2} + C\right) = 1.$$

Note that it is important that the C's are the same constant which allows us to do the cancellation above.

Sometimes we do not need to find the primitive function first to evaluate a definite integral. This could be achieved by a change of variable.

Example 3.1 *Evaluate the definite integral*

$$\int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta}\, d\theta.$$

As in Example 2.2, we can use the transformation $\theta \mapsto \frac{\pi}{2} - \theta$ to get

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta}\, d\theta = \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta + \sin\theta}\, d\theta.$$

Therefore,

$$\begin{aligned} 2I &= \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta + \sin\theta}\, d\theta + \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta}\, d\theta \\ &= \int_0^{\pi/2} \frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta}\, d\theta \\ &= \int_0^{\pi/2} 1\, d\theta = \frac{\pi}{2}. \end{aligned}$$

Therefore, $I = \pi/4$.

4 Method of Partial Fractions

This section discuss a powerful method to evaluate integrals of rational functions

$$\int \frac{p(x)}{q(x)}\, dx$$

where $p(x)$, $q(x)$ are polynomials. For example, consider the integral

$$\int \frac{1}{x(x-1)}\, dx,$$

we want to break it into two terms:

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1},$$

where A, B are some constants to be determined. Since we know how to integrate the right hand side, the only task remaining is to find A and B. Note that

$$\frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1) + Bx}{x(x-1)}.$$

Therefore, we require

$$\frac{1}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}.$$

Comparing the coefficients in the numerator, we get a system of linear equations

$$\begin{cases} A+B &= 0 \\ -A &= 1 \end{cases}$$

which gives $A = -1$ and $B = 1$. As a result,

$$\int \frac{1}{x(x-1)}\,dx = \int \left(\frac{-1}{x} + \frac{1}{x-1}\right) dx = -\ln|x| + \ln|x-1| + C.$$

Example 4.1 *If the denominator contains terms of degree two or higher, we have to include up to those order as well:*

$$\frac{x^2-2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Expanding and compare coefficients, we get $A = -2$, $B = 3$ and $C = -1$. Notice that we can also write

$$\frac{x^2-2}{x(x-1)^2} = \frac{A}{x} + \frac{Bx+C}{(x-1)^2},$$

which is actually equivalent to the form before since we can write

$$\frac{Bx+C}{(x-1)^2} = \frac{B(x-1)+(B+C)}{(x-1)^2} = \frac{B}{x-1} + \frac{B+C}{(x-1)^2}.$$

Example 4.2 *If the denominator is not given in* "product form"*, we would have to factorize it first:*

$$\frac{x^2-x+2}{2x^3+3x^2-2x} = \frac{x^2-x+2}{x(x+2)(2x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}.$$

We can expand and compare coefficients as before to get $A = 1$, $B = 2/5$ and $C = -9/5$.

Example 4.3 *If the degree of numerator is larger than the degree of denominator, we first do a long division to reduce the degree.*

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3} = 2x + \frac{A}{x-3} + \frac{B}{x+1},$$

where we get $A = 3$ and $B = 2$.

Example 4.4 *Sometimes we may not be able to factorize the denominator completely into linear factors. For example,*

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1},$$

expanding and compare we get $A = 1$, $B = -1$ and $C = 0$. Therefore,

$$\int \frac{1}{x(x^2+1)} = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + C.$$